REMARKS/ARGUMENTS

In response to the Office Action mailed April 24, 2006, Applicants amend their application and request reconsideration. In this Amendment claims 1 and 7 are cancelled and claims 13-18 are added so that claims 2-6 and 8-18 are now pending.

In this Amendment claims 2, 3, 8, and 9 are rewritten in independent form. Claims 4-6 are amended to depend from amended claim 2 and new claims 13-15 reproduce claims 4-6 but depend from newly independent claim 3. Likewise, claims 10-12 are amended to depend from newly independent claim 8 and newly added claims 16-18 reproduce claims 10-12, but depend from newly independent claim 9.

In amending claims 2 and 8, the term "fast Fourier transform" has been restored to the term originally used in the claims and patent application, namely "short-time Fourier transform". Corresponding amendments are made to the substitute specification that was supplied in response to the previous Office Action. An error was made by the undersigned in preparing the substitute specification in making that amendment because of lack of knowledge regarding the difference between a short-time Fourier transform and the more familiar fast Fourier transform. The difference is significant and well recognized in the art.

In the art, the term time-dependent Fourier transform is sometimes used as interchangeable with the term short-time Fourier transform. For example, see enclosed pages 713-722 of Oppenheim et al. *Discrete-Time Signal Processing* (1989). Attention is particularly directed to the first two lines on page 714. Also attached as evidence of the correct meaning of the term and the distinction from the fast Fourier transform, are six pages from the Wikipedia.

As well understood in the art, in a Fourier transform, a signal is analyzed by identifying sinusoidal components that, together, constitute the signal. Thus, the signal being analyzed is treated as compilation of periodic functions. When this analysis is made using a fast Fourier transform, as in the principal reference discussed further below, the first and last parts of the waveform are connected to each other.

The connection is inevitably discontinuous requiring the presence of an amplitude adjustment. As a result, false frequency components of the signal being analyzed are produced. On the other hand, as shown by the attachments, a short-time Fourier transform employs a window function that ensures that the first and last parts of waveform in the analysis have zero amplitude so that there is no discontinuity in connecting these end points. As a result, no false frequency components are generated. This effect is important in the claimed invention as defined by claims 2 and 8 and their dependent claims.

Claim 7 as formerly pending was objected to as not providing adequate antecedent basis for the reference to time-frequency transforming. In rewriting claims 8 and 9 in independent form, the part of claim 7 that is incorporated has been reformatted and amended in a way that overcomes this objection. This amendment made to overcome the objection is essentially the amendment suggested by the Examiner.

All examined claims were again rejected as not enabled by the disclosure of the patent application. Applicants again traverse this rejection based upon the previous response, which is incorporated here by reference, and the following comments.

The comments concerning language that appeared only in claim 1 regarding the "time-frequency transforming means" has been revised in amending claims 2 and 3. The language is entirely consistent with the specification, overcoming any issue as to whether the setting of time intervals is actually achieved by the time-frequency transforming means. Similar language did not appear in claim 7 although the rejection seems to encompass all claims previously pending.

The Examiner's comments reveal his clear understanding of what is disclosed in the patent application and corresponding clarifying amendments, in accordance with the Examiner's request, are made at pages 8-10 of the substitute specification. There may have been a translational error with regard to defining distinctly the time periods and time intervals that led to potential confusion in the English language specification, but did not prevent a clear understanding by the Examiner of what is

disclosed. Thus, the amendments made at pages 8-10 of the substitute specification do not alter in substance the original disclosure which meets the requirements of 35 USC 112, first paragraph, as demonstrated by the clear understanding obtained by the Examiner. The amended specification distinguishes the time intervals, as illustrated in Figure 3 of the patent application, from the much shorter time periods within the time intervals. The ion current samples are acquired during respective time periods. Further, the symbol I is now consistently used to identify current, namely ion current. The subscripts identify the time periods of sampling and the corresponding samples. As noted by the Examiner, M represents the number of such samples within a time interval.

All examined claims were rejected as unpatentable over Frankowski et al. (U.S. Patent 6,456,927, hereinafter Frankowski) in view of Malaczynski et al. (U.S. Patent 6,805,099, hereinafter Malaczynski). This rejection is respectfully traversed.

Amended claims 2 and 8 respectively describe an apparatus and a method for knocking detection in an internal combustion engine. Both the apparatus and the method employ a short-time Fourier transform in determining the frequency components of ion currents that are sampled. As described above, the short-time Fourier transform is substantially different from the fast Fourier transform. Moreover, in the environment of the invention, the use of a fast Fourier transform leads to errors because of the discontinuity in connecting the beginning and end of each waveform cycle. This error is avoided in the invention, as defined by claims 2 and 8 and their respective dependent claims, by the employment of the short-time Fourier transform.

Frankowski, but not Malaczynski, describes determining the frequency components of electrical signals sensed from an ignition system of an internal combustion engine using a fast Fourier transformation. However, there is never any discussion in either of those references of any apparatus or method that employs a short-time Fourier transform and thereby achieves the advantages of the invention as defined by claims 2 and 8 and their respective dependent claims. Every place within Frankowski that the term Fourier appears, the term appears together with the word

"fast" showing that there is no disclosure of the invention nor even any suggestion for the invention in Frankowski. Since Malaczynski is silent as to this feature of the invention, no combination of Frankowski and Malaczynski can include all of the limitations of claims 2 and 8 and their respective dependent claims. Therefore, those two patents cannot establish *prima facie* obviousness of any of those claims. Because of these differences and because of the differences previously pointed out, upon reconsideration, the rejection of claims 2, 4-6, 8, and 10-12 should be withdrawn.

Claims 3 and 9 describe an apparatus and method for knocking determination employing a Gabor wavelet, rather than the short-time Fourier transform of claims 2 and 8. Clearly, Malaczynski is relied upon in the rejection of these claims, rather than Frankowski. Frankowski never mentions the term wavelet and thus cannot describe anything similar to the invention as described by this second group of independent claims and their respective dependent claims 13-18.

Malaczynski uses the term wavelet frequently, even in the title of the patent. However, there is no reference to a Gabor wavelet in any wavelet transforms described by Malaczynski.

As shown by the attachment taken from the Wikipedia, there are an extraordinary number of different wavelet transformations. See, for example, the list of wavelets appearing at page 7 of the eight-page attachment concerning wavelets and the single page listing particular wavelet-related transforms. Malaczynski is completely silent regarding any particular wavelet transform that is contemplated in that patent. Thus, Malaczynski is practically non-enabling as to wavelet transformation.

The Gabor wavelet transform is a discrete transform, like the short-time Fourier transform of the other claims. The Gabor wavelet, like the short-time Fourier transform, employs a window function so that the generation of false frequency components is avoided, an important feature of the invention neither disclosed nor contemplated in Malaczynski. Further, Applicants respectfully note that even the attachments from the Wikipedia concerning wavelet-related transforms never make

reference to a Gabor transform which has proven particularly useful in the invention, showing that the application of the Gabor wavelet to Malaczynski is not obvious..

Since neither Frankowski nor Malaczynski describes all of the elements of the invention as defined by claims 3 and 9 and their respective dependent claims 13-18, *prima facie* obviousness has not been established with respect to any of those claims. Accordingly, upon reconsideration, those claims should be allowed.

Withdrawal of the rejection and allowance of all of claims 2-6 and 8-18 are earnestly solicited.

Respectfully submitted,

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Discrete-Time Signal Processing

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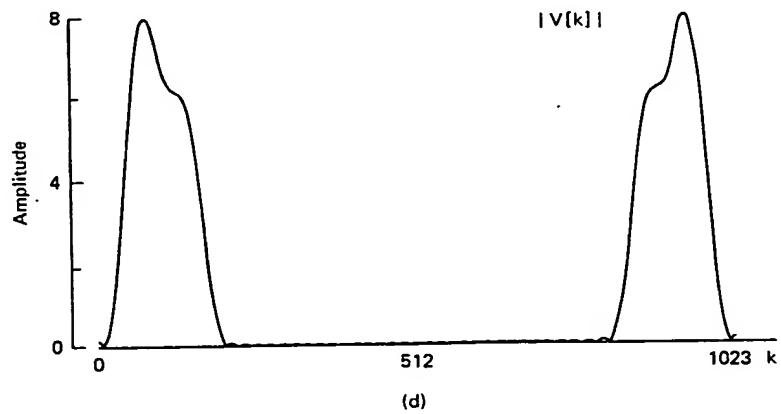


Figure 11.9 (*continued*) (d) Magnitude of DFT for $N = 1024$. (DFT values are linearly interpolated to obtain a smooth curve.)

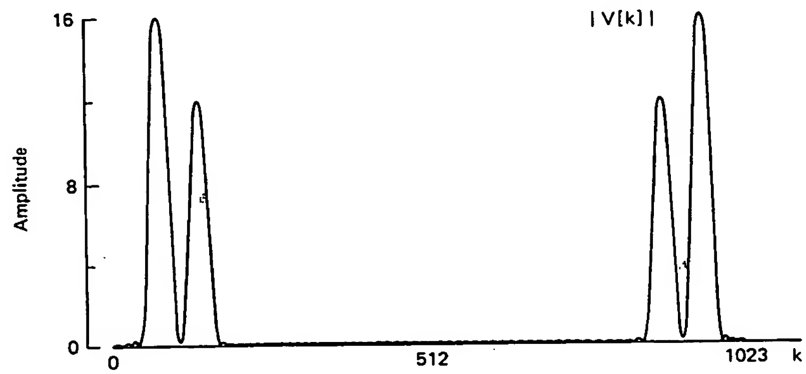


Figure 11.10 Illustration of the computation of the DFT for $N \gg L$ with linear interpolation to create a smooth curve ($N = 1024$, $L = 64$).

11.3 THE TIME-DEPENDENT FOURIER TRANSFORM

The previous section illustrated the use of the DFT for obtaining a frequency-domain representation of a signal composed of sinusoidal components. In that discussion, we assumed that the frequencies of the cosines did not change with time so that no matter how long the window, the signal properties would be the same from the beginning to the end of the window. Often, in practical applications of sinusoidal signal models, the signal properties (amplitudes, frequencies, and phases) will change with time. For example, nonstationary signal models of this type are required to describe radar, sonar, speech, and data communication signals. A single DFT estimate is not

sufficient to describe such signals, and as a result we are led to the concept of the *time-dependent Fourier transform*, also referred to as the short-time Fourier transform.†

The time-dependent Fourier transform of a signal $x[n]$ is defined as

$$X[n, \lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}, \tag{11.18}$$

where $w[n]$ is a window sequence. In the time-dependent Fourier representation, the one-dimensional sequence $x[n]$, a function of a single discrete variable, is converted into a two-dimensional function of the time variable n, which is discrete, and the frequency variable λ, which is continuous.‡ Note that the time-dependent Fourier transform is periodic in λ with period 2π, and therefore we need consider only values of λ for $0 \le \lambda < 2\pi$ or any other interval of length 2π.

Equation (11.18) can be interpreted as the Fourier transform of $x[n+m]$ as viewed through the window $w[m]$. The window has a stationary origin and as n changes, the signal slides past the window so that at each value of n, a different portion of the signal is viewed. This is depicted in Fig. 11.11 for the signal

$$x[n] = \cos(\omega_0 n^2), \quad \omega_0 = 2\pi \times 7.5 \times 10^{-6}, \tag{11.19}$$

corresponding to a linear frequency modulation (i.e. the "instantaneous frequency" is $2\omega_0 n$). As we saw in Chapter 9 in the context of the chirp transform algorithm, a signal of this type is often referred to as a linear chirp. Typically, $w[m]$ in Eq. (11.18) has finite length around $m = 0$ so that $X[n, \lambda)$ displays the frequency characteristics of the signal around time n. As an example, in Fig. 11.12 we show a display of the magnitude of the time-dependent Fourier transform of the signal of Eq. (11.19) and Fig. 11.11 with $w[m]$ a Hamming window of length 400. In this display, referred to as a *spectrogram*, the vertical dimension is frequency (λ) and the horizontal dimension is time (n). The magnitude of the time-dependent Fourier transform is represented by the darkness of the markings. In Fig. 11.12, the linear progression of the frequency with time is clear.

Since $X[n, \lambda)$ is the discrete-time Fourier transform of $x[n+m]w[m]$, the time-dependent Fourier transform is invertible if the window has at least one nonzero sample. Specifically, from the Fourier transform synthesis equation (2.112),

$$x[n+m]w[m] = \frac{1}{2\pi}\int_0^{2\pi} X[n, \lambda)e^{j\lambda m}\, d\lambda, \qquad -\infty < m < \infty, \tag{11.20}$$

from which it follows that

$$x[n] = \frac{1}{2\pi w[0]}\int_0^{2\pi} X[n, \lambda)\, d\lambda \tag{11.21}$$

† Further discussion of the time-dependent Fourier transform can be found in a variety of references, including Allen and Rabiner (1977), Rabiner and Schafer (1978), Crochiere and Rabiner (1983), and Nawab and Quatieri (1988).

‡ We denote the frequency variable of the time-dependent Fourier transform by λ to maintain a distinction from the frequency variable of the conventional discrete-time Fourier transform, which will be denoted ω. We use the mixed bracket-parenthesis notation $X[n, \lambda)$ as a reminder that n is a discrete variable and λ a continuous variable.

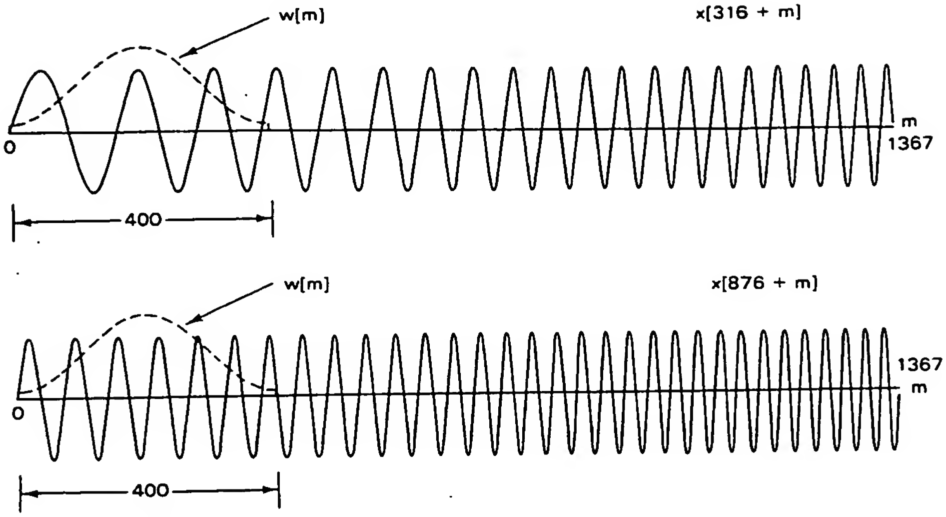


Figure 11.11 Two segments of the linear chirp signal $x[n] = \cos(2\pi \times 7.5 \times 10^{-06})n^2$ with the window superimposed. $X[n,\lambda)$ at $n = 316$ is the discrete-time Fourier transform of the top trace multiplied by the window. $X[876, \lambda)$ is the discrete-time Fourier transform of the bottom trace multiplied by the window.

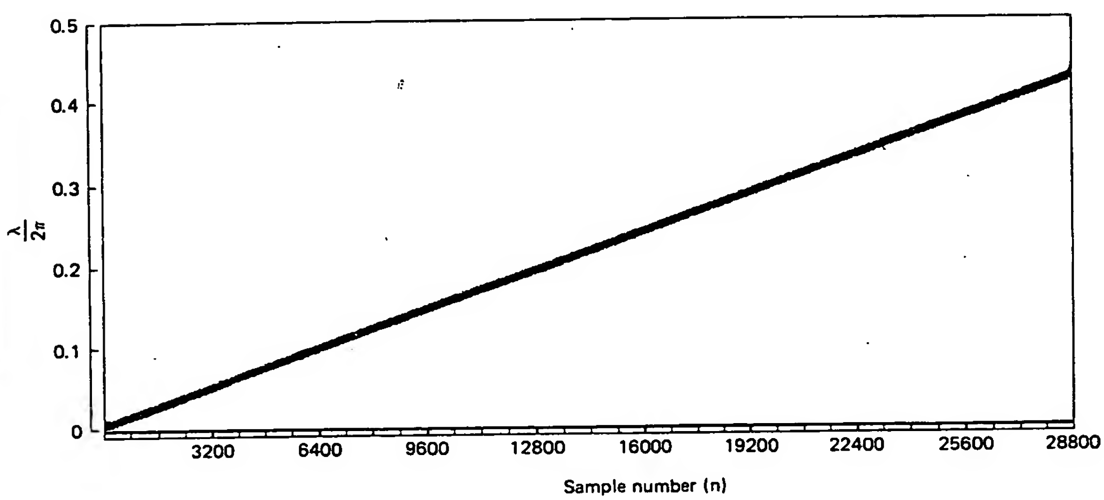


Figure 11.12 The magnitude of the time-dependent Fourier transform of $x[n] = \cos(2\pi \times 7.5 \times 10^{-6})n^2$ using a Hamming window of length 400.

if $w[0] \neq 0$.† Clearly not just the single sample $x[n]$ but all of the samples that are multiplied by nonzero samples of the window can be recovered in a similar manner using Eq. (11.20).

A rearrangement of the sum in Eq. (11.18) leads to another useful interpretation of the time-dependent Fourier transform. If we make the substitution $m' = n + m$ in Eq. (11.18), then $X[n, \lambda)$ can be written as

$$X[n, \lambda) = \sum_{m'=-\infty}^{\infty} x[m']w[-(n - m')]e^{j\lambda(n-m')}. \tag{11.22}$$

Equation (11.22) can be interpreted as the convolution

$$X[n, \lambda) = x[n] * h_\lambda[n], \tag{11.23a}$$

where

$$h_\lambda[n] = w[-n]e^{j\lambda n}. \tag{11.23b}$$

From Eq. (11.23a) we see that the time-dependent Fourier transform as a function of n with λ fixed can be interpreted as the output of a linear time-invariant filter with impulse response $h_\lambda[n]$, or, equivalently, with frequency response

$$H_\lambda(e^{j\omega}) = W(e^{j(\lambda-\omega)}). \tag{11.24}$$

In general a window that is nonzero for positive time will be called a *noncausal window* since the computation of $X[n, \lambda)$ using Eq. (11.18) requires samples that *follow* sample n in the sequence. Equivalently, in the linear filtering interpretation, the impulse response $h_\lambda[n] = w[-n]e^{j\lambda n}$ is noncausal.

In the definition of Eq. (11.18), the time origin of the window is held fixed and the signal is shifted past the interval of support of the window. This effectively redefines the time origin for Fourier analysis to be at sample n of the signal. Another possibility is to shift the window as n changes, keeping the time origin for Fourier analysis fixed at the original time origin of the signal. This leads to a definition for the time-dependent Fourier transform of the form

$$\check{X}[n, \lambda) = \sum_{m=-\infty}^{\infty} x[m]w[m - n]e^{-j\lambda m}. \tag{11.25}$$

The relationship between the definitions of Eqs. (11.18) and (11.25) is easily shown to be

$$\check{X}[n, \lambda) = e^{-j\lambda n}X[n, \lambda) \tag{11.26}$$

The definition of Eq. (11.18) is particularly convenient when we consider using the DFT to obtain samples in λ of the time-dependent Fourier transform since if $w[m]$ is of finite length in the range $0 \leq m \leq (L - 1)$, then so is $x[n + m]w[m]$. On the other hand, the definition of Eq. (11.25) has some advantages for the interpretation of Fourier analysis in terms of filter banks. Since our primary interest is in applications of the DFT, we will base our discussion on Eq. (11.18).

† Since $X[n, \lambda)$ is periodic in λ with period 2π, the integration in Eqs. (11.20) and (11.21) can be over any interval of length 2π.

11.3.1 The Effect of the Window

The primary purpose of the window in the time-dependent Fourier transform is to limit the extent of the sequence to be transformed so that the spectral characteristics are reasonably stationary over the duration of the window. The more rapidly the signal characteristics change, the shorter the window should be. As we saw in Section 11.2, as the window becomes shorter, frequency resolution decreases. The same effect is true, of course, for $X[n, \lambda)$. On the other hand, as the window length decreases, the ability to resolve changes with time increases. Consequently, the choice of window length becomes a tradeoff between frequency resolution and time resolution.

The effect of the window on the properties of the time-dependent Fourier transform can be seen by assuming that the signal $x[n]$ has a conventional discrete-time Fourier transform $X(e^{j\omega})$. First let us assume that the window is unity for all m; i.e., assume that there is no window at all. Then from Eq. (11.18),

$$X[n, \lambda) = X(e^{j\lambda})e^{j\lambda n}. \tag{11.27}$$

Of course, a typical window for spectrum analysis tapers to zero so as to select only a portion of the signal for analysis. As discussed in Section 11.2, the length and shape of the window are chosen so that the Fourier transform of the window is narrow in frequency compared with changes in the Fourier transform of the signal. The Fourier transform of a typical window is depicted in Fig. 11.13(a).

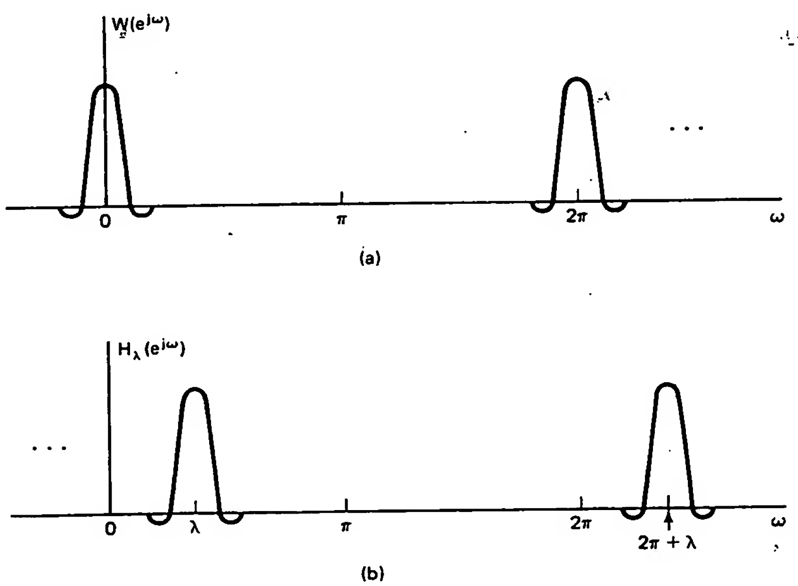


Figure 11.13 (a) Fourier transform of window in time-dependent Fourier analysis. (b) Equivalent bandpass filter for time-dependent Fourier analysis.

If we consider the time-dependent Fourier transform for fixed n, then it follows from the properties of Fourier transforms that

$$X[n, \lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{j\theta n}X(e^{j\theta})W(e^{j(\lambda-\theta)})\,d\theta, \qquad (11.28)$$

i.e., the Fourier transform of the shifted signal is convolved with the Fourier transform of the window. This is similar to Eq. (11.2) except that in Eq. (11.2) we assumed that the signal was not successively shifted relative to the window. Here we compute a Fourier transform for each value of n. In Section 11.2 we saw that the ability to resolve two narrowband signal components depends on the width of the mainlobe of the Fourier transform of the window, while the degree of leakage of one component into the vicinity of the other depends on the relative sidelobe amplitude.

In the linear filtering interpretation of Eqs. (11.23) and (11.24), $W(e^{j\omega})$ typically has the lowpass characteristics depicted in Fig. 11.13(a), and consequently $H_\lambda(e^{j\omega})$ is a bandpass filter whose passband is centered at $\omega = \lambda$, as depicted in Fig. 11.13(b). Clearly, the width of the passband of this filter is approximately equal to the width of the mainlobe of the Fourier transform of the window. The degree of rejection of adjacent frequency components depends on the relative sidelobe amplitude.

The preceding discussion suggests that if we are using the time-dependent Fourier transform to obtain a time-dependent estimate of the frequency spectrum of a signal, it is desirable to taper the window to lower the sidelobes and to use as long a window as feasible to improve the frequency resolution. We will consider some examples in Section 11.5. However, before doing so, we first discuss the use of the DFT in explicitly evaluating the time-dependent Fourier transform.

11.3.2 Sampling in Time and Frequency

Explicit computation of $X[n, \lambda)$ can be done only at a finite set of values of λ, corresponding to sampling the time-dependent Fourier transform in the frequency variable domain. Just as finite-length signals can be exactly represented through samples of the discrete-time Fourier transform, signals of indeterminate length can be represented through samples of the time-dependent Fourier transform if the window in Eq. (11.18) has finite length. As an example, suppose that the window has length L with samples beginning at $m = 0$, i.e.,

$$w[m] = 0 \qquad \text{outside the interval } 0 \le m \le L-1. \qquad (11.29)$$

If we sample $X[n, \lambda)$ at N equally spaced frequencies $\lambda_k = 2\pi k/N$, with $N \ge L$, then we can still recover the original sequence from the sampled time-dependent Fourier transform. Specifically, if we define $X[n, k]$ to be

$$X[n, k] = X[n, 2\pi k/N) = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}, \qquad 0 \le k \le N-1, \quad (11.30)$$

Short-time Fourier transform

From Wikipedia, the free encyclopedia

The **short-time Fourier transform** (STFT), or alternatively **short-term Fourier transform**, is a Fourier-related transform used to determine the sinusoidal frequency and phase content of local sections of a signal as it changes over time.

Simply described, in the continuous-time case, the function to be transformed is multiplied by a window function which is nonzero for only a short period of time. The Fourier transform (a one-dimensional function) of the resulting signal is taken as the window is slid along the time axis, resulting in a two-dimensional representation of the signal. Mathematically, this is written as:

$$\mathbf{STFT}\{x(\ )\} \equiv X(\tau,\omega) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-j\omega t}\,dt$$

where $w(t)$ is the window function, commonly a Hann window or gaussian "hill" centered around zero, and $x(t)$ is the signal to be transformed. $X(\tau,\omega)$ is essentially the Fourier Transform of $x(t)w(t-\tau)$, a complex function representing the phase and magnitude of the signal over time and frequency. Often phase unwrapping is employed along either or both the time axis, τ and frequency axis, ω, to suppress any jump discontinuity of the phase result of the STFT. The time index τ is normally considered to be "*slow*" time and usually not expressed in as high resolution as time t.

In the discrete time case, the data to be transformed could be broken up into chunks or frames (which usually overlap each other). Each chunk is Fourier transformed, and the complex result is added to a matrix, which records magnitude and phase for each point in time and frequency. This can be expressed as:

$$\mathbf{STFT}\{x[\ ]\} \equiv X(m,\omega) = \sum_{n=-\infty}^{\infty} x[n]w[n-m]e^{-j\omega n}$$

likewise, with signal $x[n]$ and window $w[n]$. In this case, m is discrete and ω is continuous, but in most typical applications the STFT is performed on a computer using the Fast Fourier Transform, so both variables are discrete and quantized. Again, the discrete-time index m is normally considered to be "slow" time and usually not expressed in as high resolution as time n.

The magnitude squared of the STFT yields the spectrogram of the function:

$$\text{spectrogram}\{x(\ )\} \equiv |X(\tau,\omega)|^2$$

Contents

Inverse STFT

The STFT is invertible, that is, the original signal can be recovered from the transform by the Inverse STFT.

Continuous-time STFT

Given the width and definition of the window function *w(t)*, we initially require the height of the window function to be scaled so that

$$\int_{-\infty}^{\infty} w(\tau)\, d\tau = 1 .$$

It easily follows that

$$\int_{-\infty}^{\infty} w(t-\tau)\, d\tau = 1 \quad \forall\ t$$

and

$$x(t) = x(t) \int_{-\infty}^{\infty} w(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t) w(t-\tau)\, d\tau .$$

The continuous Fourier Transform is

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt .$$

Substituting $x(t)$ from above:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) w(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) w(t-\tau)\, e^{-j\omega t}\, d\tau\, dt \end{aligned}$$

Swapping order of integration:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) w(t-\tau)\, e^{-j\omega t}\, dt\, d\tau \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) w(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau \\ &= \int_{-\infty}^{\infty} X(\tau, \omega)\, d\tau \end{aligned}$$

So the Fourier Transform can be seen as a sort of phase coherent sum of all of the STFTs of $x(t)$. Since the inverse Fourier Transform is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{+j\omega t} \, d\omega \ ,$$

then $x(t)$ can be recovered from $X(\tau,\omega)$ as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\tau, \omega) e^{+j\omega t} \, d\tau \, d\omega \ .$$

or

$$x(t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau, \omega) e^{+j\omega t} \, d\omega \right] \, d\tau \ .$$

It can be seen, comparing to above that windowed "grain" or "wavelet" of $x(t)$ is

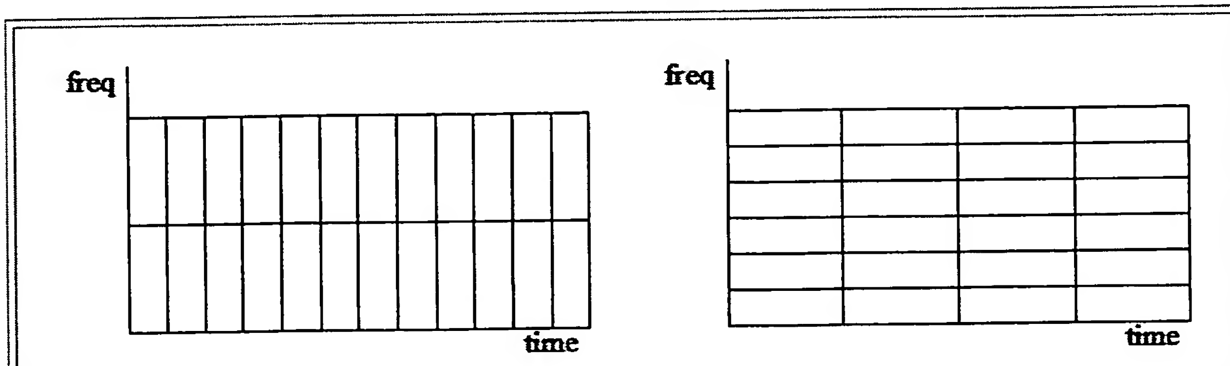
$$x(t) w(t - \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau, \omega) e^{+j\omega t} \, d\omega \ ,$$

the inverse Fourier Transform of $X(\tau,\omega)$ for τ fixed.

Discrete-time STFT

Resolution issues

One of the downfalls of the STFT is that it has a fixed resolution. The width of the windowing function relates to the how the signal is represented — it determines whether there is good frequency resolution (frequency components close together can be separated) or good time resolution (the time at which frequencies change). A wide window gives better frequency resolution but poor time resolution. A narrower window (said to be compactly supported) gives good time resolution but poor frequency resolution. These are called narrowband and wideband transforms, respectively.



Comparison of STFT resolution. Left has better time resolution, and right has better frequency resolution

This is one of the reasons for the creation of the wavelet transform (or multiresolution analysis in general), which can give good time resolution for high-frequency events, and good frequency resolution for low-frequency events, which is the type of analysis best suited for many real signals.

This property is related to the Heisenberg uncertainty principle, but it is not a direct relationship. The product of the standard deviation in time and frequency is limited. The boundary of the uncertainty principle (best

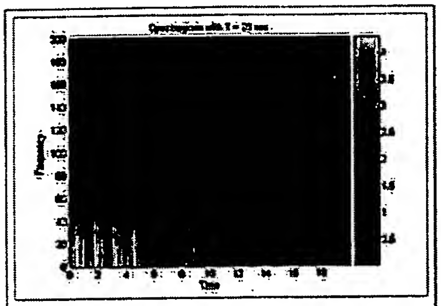
simultaneous resolution of both) is reached with a gaussian window function.

Example

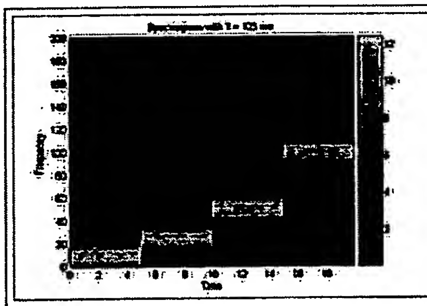
Using the following sample signal that has 4 frequencies (10, 25, 50, 100 Hz) that are never present at the same time:

$x(t) = \cos(2\pi 10t)$ for $0 \leq t < 0.1s$
$x(t) = \cos(2\pi 25t)$ for $0.1s \leq t < 0.2s$
$x(t) = \cos(2\pi 50t)$ for $0.2s \leq t < 0.3s$
$x(t) = \cos(2\pi 100t)$ for $0.3s \leq t < 0.4s$

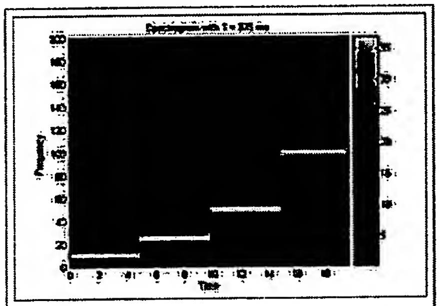
The following spectrograms were produced:



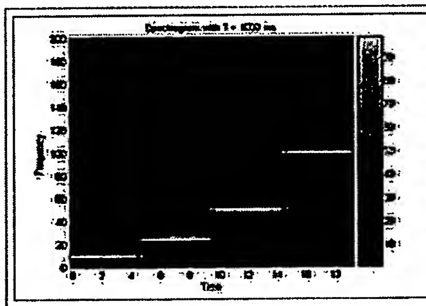
25 ms window



125 ms window



375 ms window



1000 ms window

The 25 ms window allows the exact time the signals change to be identified but the frequencies are difficult to identify. At the other end of the scale, the 1000 ms window allows the frequencies to be precisely seen but the time between frequency changes is blurred.

Explanation

It can also be explained with reference to the sampling and Nyquist frequency.

Take a window of N samples from an arbitrary real-valued signal at sampling rate f_s . Taking the Fourier transform produces N coefficients. Of these coefficients only the first $N/2$ are useful (the others are just a mirror image as this is a real valued signal).

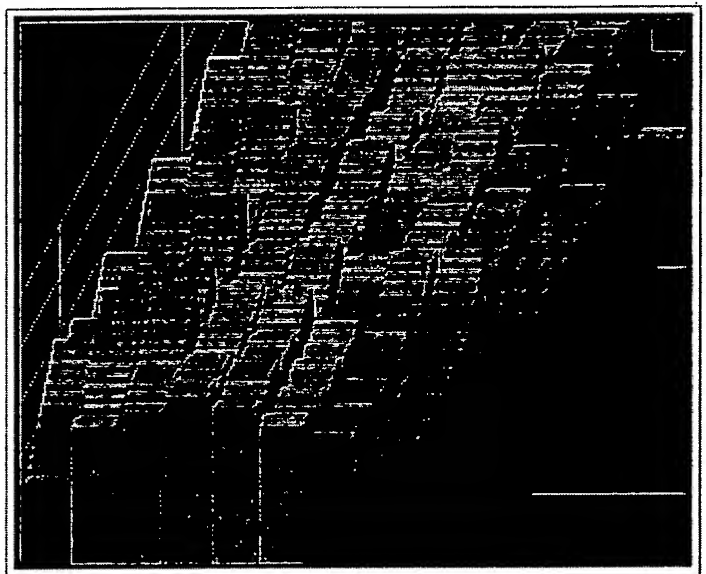
These $N/2$ coefficients represent the frequencies 0 to $f_s/2$ (Nyquist), meaning that two consecutive coefficients are spaced apart by

$$\frac{f_s/2}{(N/2)-1}\ \mathrm{Hz}$$

For large N this approximates to $\frac{f_s}{N}$

To increase the frequency resolution of the window the frequency spacing of the coefficients needs to be reduced. There are only two variables but decreasing f_s (and keeping N constant) will cause the window size to increase — since there are now less samples per unit time. The other alternative is to increase N, but this again causes the window size to increase. So any attempt to increase the frequency resolution causes a larger window size and therefore a reduction in time resolution — and vice-versa.

Application



A STFT being used to analyse a song across time.

STFTs as well as standard fourier transforms and other tools are frequently used to analyse music. The image shows frequency on the horizontal axis, with the lowest frequencies at left, and the highest at the right. The height of each bar (augmented by color) is the amplitude of the frequencies within that band. And the depth dimension is time, where each new bar was a separate distinct transform. Audio engineers use this kind of visual to gain information about an audio sample, for example to locate the frequencies of specific noises (especially when used with greater frequency resolution) or to find frequencies which may be more or less resonant in the space where the signal was recorded. This information can be used for Equalization or tuning other audio effects.

See also

Other time-frequency transforms

- wavelet transform
- chirplet transform
- fractional Fourier transform

External links

- DiscreteTFDs -- software for computing the short-time Fourier transform and other time-frequency distributions (http://tfd.sourceforge.net/)
- Singular Spectral Analysis - MultiTaper Method Toolkit (http://www.atmos.ucla.edu/tcd/ssa/) - a software program to analyze short, noisy time series. (Free download as of 18 May, 2005)

Retrieved from "http://en.wikipedia.org/wiki/Short-time_Fourier_transform"

Categories: Signal processing | Transforms

List of wavelet-related transforms

From Wikipedia, the free encyclopedia

A list of wavelet related transforms:

- Continuous wavelet transform (CWT)
- Multiresolution analysis (MRA)
- Discrete wavelet transform (DWT)
- Fast wavelet transform (FWT)
- Complex wavelet transform
- Non or undecimated wavelet transform, the downsampling is omitted
- Newland transform, an orthonormal basis of wavelets is formed from appropriately constructed top-hat filters in frequency space
- Wavelet packet decomposition (WPD), detail coefficients are decomposed and a variable tree can be formed
- Stationary wavelet transform, no downsampling and the filters at each level are different
- e-decimated discrete wavelet transform, depends on if the even or odd coefficients are selected in the downsampling
- Second generation wavelet transform (SGWT), filters and wavelets are not created in the frequency domain
- Dual-tree complex wavelet transform (DTCWT), two trees are used for decomposion to produce the real and complex coefficients

Retrieved from "http://en.wikipedia.org/wiki/List_of_wavelet-related_transforms"

Categories: Wavelets | Mathematics-related lists

- This page was last modified 15:30, 9 April 2006.

Wavelet

From Wikipedia, the free encyclopedia

In mathematics, **wavelets**, **wavelet analysis**, and the **wavelet transform** refers to the representation of a signal in terms of a finite length or fast decaying oscillating waveform (known as the **mother wavelet**). This waveform is scaled and translated to match the input signal. In formal terms, this representation is a wavelet series, which is the coordinate representation of a square integrable function with respect to a complete, orthonormal set of basis functions for the Hilbert space of square integrable functions. Note that the wavelets in the JPEG2000 standard are biorthogonal wavelets, that is, the coordinates in the wavelet series are computed with a different, dual set of basis functions.

Contents

Overview

The word *wavelet* is due to Morlet and Grossman in the early 1980s. They used the French word *ondelette* - meaning "small wave". A little later it was transformed into English by translating "onde" into "wave" - giving wavelet. Wavelet transforms are broadly classified into the discrete wavelet transform (DWT) and the continuous wavelet transform (CWT). The principal difference between the two is the continuous transform operates over every possible scale and translation whereas the discrete uses a specific subset of all scale and translation values.

Using wavelet theory

Wavelet theory is applicable to several other subjects. All wavelet transforms may be considered to be forms of time-frequency representation and are, therefore, related to the subject of harmonic analysis. Almost all practically useful *discrete wavelet transforms* make use of filterbanks containing finite impulse response filters. The wavelets forming a CWT are subject to Heisenberg's uncertainty principle and, equivalently, discrete wavelet bases may be considered in the context of other forms of the uncertainty principle.

Outline of the wavelet theory

Wavelet transforms are broadly divided into three classes, the continuous wavelet transform, the discretised wavelet transform and multiresolution-based wavelet transforms.

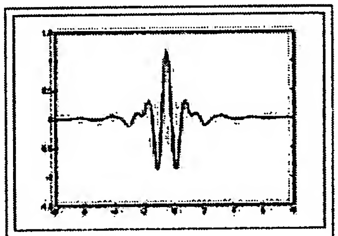
Continuous wavelet transforms

In the continuous wavelet transform, a given signal of finite energy is projected on a continuous family of frequency bands (or similar subspaces of the function space $L^2(\mathbb{R})$), for instance on every frequency band of the form $[f,2f]$ for all positive frequencies $f>0$. By a suitable integration over all the thus obtained frequency components one can reconstruct the original signal.

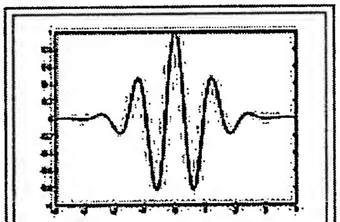
The frequency bands or subspaces are scaled versions of a subspace at scale 1. This subspace in turn is in most situations generated by the shifts of one generating function $\psi \in L^2(\mathbb{R})$, the *mother wavelet*. For the example of the scale one frequency band [1,2] this function is

$$\psi(t) = 2\,\mathrm{sinc}(2t) - \mathrm{sinc}(t) = \frac{\sin(2\pi t) - \sin(\pi t)}{\pi t}$$

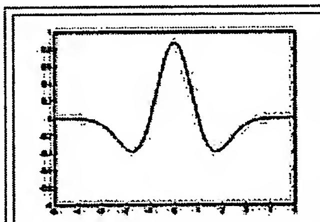
with the (normalized) sinc function. Other example mother wavelets are:



Meyer



Morlet



Mexican Hat

The subspace of scale *a* or frequency band $[1/a,\ 2/a]$ is generated by the functions (sometimes called *baby wavelets*)

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right),$$

where *a* is positive and defines the scale and *b* is any real number and defines the shift. The pair *(a,b)* defines a point in the upper halfplane $\mathbb{R}_+ \times \mathbb{R}$.

The projection of a function *x* onto the subspace of scale *a* has then the form

$$x_a(t) = \int_{\mathbb{R}} WT_\psi\{x\}(a,b) \cdot \psi_{a,b}(t)\, db$$

with *wavelet coefficients*

$$WT_\phi\{x\}(a,b) = \langle x, \psi_{a,b}\rangle = \int_{\mathbb{R}} x(t)\overline{\psi_{a,b}(t)}\,dt\,.$$

For the analysis of the signal x, one can assemble the wavelet coefficients into a scaleogram of the signal.

Discretized wavelet transforms

It is computationally impossible to analyze a signal using all wavelet coefficients. So one may wonder if it is sufficient to pick a discrete subset of the upper halfplane to be able to reconstruct a signal from the corresponding wavelet coefficients. One such system is the affine system for some real parameters $a>1$, $b>0$. The corresponding discrete subset of the halfplane consists of all the points $(a^m, n\,a^m b)$ with integers $m, n \in \mathbb{Z}$. The corresponding *baby wavelets* are now given as

$$\psi_{m,n}(t) = a^{-m/2}\psi(a^{-m}t - nb).$$

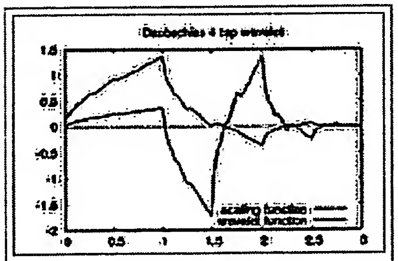
A sufficient condition for the reconstruction of any signal x of finite energy by the formula

$$x(t) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle x,\ \psi_{m,n}\rangle \cdot \psi_{m,n}(t)$$

is that the functions $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$ form a tight frame of $L^2(\mathbb{R})$.

MRA based discrete wavelet transforms

In each instance of the disretised wavelet transform, there are only a finite number of wavelet coefficients for each bounded rectangular region in the upper halfplane. Still, each coefficient requires the evaluation of an integral. To avoid this numerical complexity one needs one auxiliary function, the *father wavelet* $\phi \in L^2(\mathbb{R})$. Further, one has to restrict a to be an integer number. A typical choice is $a=2$ and $b=1$. The most famous pair of father and mother wavelets is the Daubechies 4 tap wavelet.



D4 wavelet

From the mother and father wavelets one constructs the subspaces

$$V_m = \operatorname{span}(\phi_{m,n} : n \in \mathbb{Z}),\ \text{where } \varphi_{m,n}(t) = 2^{-m/2}\varphi(2^{-m}t - n)$$

and

$$W_m = \operatorname{span}(\psi_{m,n} : n \in \mathbb{Z}),\ \text{where } \psi_{m,n}(t) = 2^{-m/2}\psi(2^{-m}t - n).$$

From these one requires that the sequence

$$\{0\} \subset \cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \subset L^2(\mathbb{R})$$

forms a multiresolution analysis of $L^2(\mathbb{R})$ and that the subspaces $\dots, W_1, W_0, W_{-1}, \dots\dots$ are the orthogonal "differences" of the above sequence, that is, W_m ist the orthogonal complement of V_m inside the subspace V_{m-1}. In analogy to the sampling theorem one may conclude that the space V_m with sampling distance 2^m more or less covers the frequency baseband from *0* to 2^{-m-1}. As orthogonal complement, W_m roughly covers the band $[2^{-m-1}, 2^{-m}]$.

From those inclusions and orthogonality relations follows the existence of sequences $h = \{h_n\}_{n\in\mathbb{Z}}$ and $g = \{g_n\}_{n\in\mathbb{Z}}$ that satisfy the identities

$$h_n = \langle \phi_{0,0}, \phi_{1,n} \rangle \text{ and } \phi(t) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \phi(2t-n)$$

and

$$g_n = \langle \psi_{0,0}, \phi_{1,n} \rangle \text{ and } \psi(t) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n \phi(2t-n).$$

The second identity of the first pair is a refinement equation for the father wavelet φ. Both pairs of identities form the basis for the algorithm of the fast wavelet transform.

Mother wavelet

For practical applications one prefers for efficiency reasons continuously differentiable functions with compact support as mother (prototype) wavelet (functions). However, to satisfy analytical requirements (in the continuous WT) and in general for theoretical reasons one chooses the wavelet functions from a subspace of the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This is the space of measurable functions that are both absolutely and square integrable:

$$\int_{-\infty}^{\infty} |\psi(t)|\, dt < \infty \text{ and } \int_{-\infty}^{\infty} |\psi(t)|^2\, dt < \infty.$$

Being in this space ensures that one can formulate the conditions of zero mean and square norm one:

$\int_{-\infty}^{\infty} \psi(t)\, dt = 0$ is the condition for zero mean, and
$\int_{-\infty}^{\infty} |\psi(t)|^2\, dt = 1$ is the condition for square norm one.

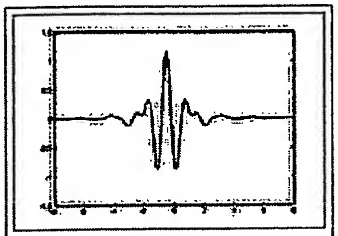
For ψ to be a wavelet for the continuous wavelet transform (see there for exact statement), the mother wavelet must satisfy an admissibility criterion (loosely speaking, a kind of half-differentiability) in order to get a stably invertible transform.

For the discrete wavelet transform, one needs at least the condition that the wavelet series is a representation of the identity in the space $L^2(\mathbb{R})$. Most constructions of discrete WT make use of the multiresolution analysis, which defines the wavelet by a scaling function. This scaling function itself is solution to a functional equation.

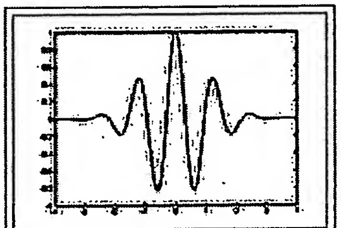
In most situations it is useful to restrict ψ to be a continuous function with a higher number M of vanishing moments, i.e. for all integer $m<M$

$$\int_{-\infty}^{\infty} t^m\,\psi(t)\,dt = 0$$

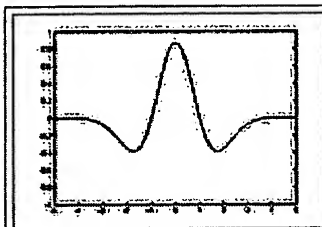
Some example mother wavelets are:



Meyer



Morlet



Mexican Hat

The mother wavelet is scaled (or dilated) by a factor of a and translated (or shifted) by a factor of b to give (under Morlet's original formulation):

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right).$$

For the continuous WT, the pair (a,b) varies over the full half-plane $\mathbb{R}_+ \times \mathbb{R}$; for the discrete WT this pair varies over a discrete subset of it, which is also called *affine group*.

These functions are often incorrectly referred to as the basis functions of the (continuous) transform. In fact, as in the continuous Fourier transform, there is no basis in the continuous wavelet transform. Time-frequency interpretation uses a subtly different formulation (after Delprat).

Comparisons with Fourier

The wavelet transform is often compared with the Fourier transform, in which signals are represented as a sum of sinusoids. The main difference is that wavelets are localized in both time and frequency whereas the standard Fourier transform is only localized in frequency. The Short-time Fourier transform (STFT) is also time and frequency localized but there are issues with the frequency time resolution and wavelets often give a better signal representation using Multiresolution analysis.

The discrete wavelet transform is also less computationally complex, taking O(N) time as compared to O(N log N) for the fast Fourier transform (N is the data size).

Definition of a wavelet

There are a number of ways of defining a wavelet (or a wavelet family).

Scaling filter

The wavelet is entirely defined by the scaling filter g - a low-pass finite impulse response (FIR) filter of length $2N$ and sum 1. In biorthogonal wavelets, separate decomposition and reconstruction filters are defined.

For analysis the high pass filter is calculated as the QMF of the low pass, and reconstruction filters the time reverse of the decomposition.

Daubechies and Symlet wavelets can be defined by the scaling filter.

Scaling function

Wavelets are defined by the wavelet function $\psi(t)$ (i.e. the mother wavelet) and scaling function $\varphi(t)$ (also called father wavelet) in the time domain.

The wavelet function is in effect a band-pass filter and scaling it for each level halves its bandwidth. This creates the problem that in order to cover the entire spectrum an infinite number of levels would be required. The scaling function filters the lowest level of the transform and ensures all the spectrum is covered. See [1] (http://perso.wanadoo.fr/polyvalens/clemens/wavelets/wavelets.html#note7) for a detailed explanation.

For a wavelet with compact support, $\varphi(t)$ can be considered finite in length and is equivalent to the scaling filter g.

Meyer wavelets can be defined by scaling functions

Wavelet function

The wavelet only has a time domain representation as the wavelet function $\psi(t)$.

Mexican hat wavelets can be defined by a wavelet function.

Applications

Generally, the DWT is used for source coding whereas the CWT is used for signal analysis. Consequently, the DWT is commonly used in engineering and computer science and the CWT is most often used in scientific research. Wavelet transforms are now being adopted for a vast number of different applications, often replacing the conventional Fourier transform. Many areas of physics have seen this paradigm shift, including molecular dynamics, ab initio calculations, astrophysics, density-matrix localisation, seismic geophysics, optics, turbulence and quantum mechanics. Other areas seeing this change have been image processing, blood-pressure, heart-rate and ECG analyses, DNA analysis, protein analysis, climatology, general signal processing, speech recognition, computer graphics and multifractal analysis. In computer vision and image processing, the notion of scale-space representation and Gaussian derivative operators is regarded as a canonical multi-scale representation.

One use of wavelets is in data compression. Like several other transforms, the wavelet transform can be used to transform raw data (like images), then encode the transformed data, resulting in effective compression. JPEG 2000 is an image standard that uses wavelets. For details see wavelet compression.

History

The development of wavelets can be linked to several separate trains of thought, starting with Haar's work in the early 20th century. Notable contributions to wavelet theory can be attributed to Goupillaud, Grossman and Morlet's formulation of what is now known as the CWT (1982), Strömberg's early work on discrete wavelets (1983), Daubechies' orthogonal wavelets with compact support (1988), Mallat's multiresolution framework (1989), Delprat's time-frequency interpretation of the CWT (1991), Newland's Harmonic wavelet transform and many others since.

Time line

- First wavelet (Haar wavelet) by Alfred Haar (1909)
- Since the 1950s: Jean Morlet and Alex Grossman
- Since the 1980s: Yves Meyer, Stéphane Mallat, Ingrid Daubechies, Ronald Coifman, Victor Wickerhauser, Nick Kingsbury

Wavelet transforms

There are a large number of wavelet transforms each suitable for different applications. For a full list see list of wavelet-related transforms but the common ones are listed below:

- Continuous wavelet transform (CWT)
- Discrete wavelet transform (DWT)
- Fast wavelet transform (FWT)
- Wavelet packet decomposition (WPD)
- Stationary wavelet transform (SWT)

List of wavelets

Discrete wavelets

- Beylkin (18)
- Coiflet (6, 12, 18, 24, 30)
- Daubechies wavelet (2, 4, 6, 8, 10, 12, 14, 16, 18, 20)
- Cohen-Daubechies-Feauveau wavelet (Sometimes referred to as Daubechies biorthogonal, bior44=CDF9/7)
- Haar wavelet
- Symmlet
- Complex wavelet transform

Continuous wavelets

- Mexican hat wavelet
- Hermitian wavelet
- Hermitian hat wavelet
- Complex mexican hat wavelet
- Morlet wavelet
- Modified Morlet wavelet
- Hilbert-Hermitian wavelet

See also

- Filter banks
- Scale space
- Ultra wideband radio- transmits wavelets.
- short-time Fourier transform
- chirplet transform
- fractional Fourier transform

References

- Paul S. Addison, *The Illustrated Wavelet Transform Handbook*, Institute of Physics, 2002, ISBN 0-

7503-0692-0

- Ingrid Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, 1992, ISBN 0-89871-274-2
- P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, 1993, ISBN 0-13-605718-7
- Mladen Victor Wickerhauser, *Adapted Wavelet Analysis From Theory to Software*, A K Peters Ltd, 1994, ISBN 1-56881-041-5
- Gerald Kaiser, *A Friendly Guide to Wavelets*, Birkhauser, 1994, ISBN 0-8176-3711-7

External links

- Wavelets and Turbulence (http://wavelets.ens.fr/)
- Wavelets made Simple (http://www.ee.ryerson.ca/~jsantarc/html/theory.html)
- Wavelet Digest (http://www.wavelet.org/)
- Amaras Wavelet Page (http://www.amara.com/current/wavelet.html)
- Wavelet Posting Board (http://ondelette.com/indexen.html)
- The Wavelet Tutorial by Polikar (http://users.rowan.edu/~polikar/WAVELETS/WTtutorial.html)
- OpenSource Wavelet C Code (http://herbert.the-little-red-haired-girl.org/en/software/wavelet/)
- An Introduction to Wavelets (http://www.amara.com/IEEEwave/IEEEwavelet.html)
- Wavelet-based time-frequency analysis in Mathematica (http://www.ffconsultancy.com/products/CWT/HTML/tutorial.html) and example analyses from physics, biology, engineering, bioinformatics and finance.
- Wavelets for Kids (PDF file) (http://www.isye.gatech.edu/~brani/wp/kidsA.pdf) (introductory)
- Link collection about wavelets (http://www.cosy.sbg.ac.at/~uhl/wav.html)
- List of Wavelet resources, libraries and source codes (http://www.compression-links.info/Wavelets)
- Wavelet forums (French) (http://www.ondelette.com/index.html) Wavelet forum (English) (http://www.ondelette.com/indexen.html)
- "Biorthogonal sinc wavelets" (http://users.atw.hu/uranium/image_codec_doc/biosinc_20051204.pdf)
- Gerald Kaiser's acoustic and electromagnetic wavelets (http://www.wavelets.com/center.php)
- A really friendly guide to wavelets (http://perso.wanadoo.fr/polyvalens/clemens/wavelets/wavelets.html)

Categories: Numerical analysis | Signal processing | Image processing | Wavelets